ECE 307 – Techniques for Engineering Decisions

14. Simulation

George Gross

Department of Electrical and Computer Engineering

University of Illinois at Urbana-Champaign

SIMULATION

- **Simulation provides a *systematic* approach for dealing with uncertainty by "*flipping a coin*" or "*rolling a die*" to represent the outcome or realization of each uncertain event**
- **In many real world situations, simulation may be the *only viable means* to quantitatively deal with a problem under uncertainty**
- **Effective simulation requires implementation of appropriate approximations at many and, some-times, at possibly every stage of the problem**

SIMULATION EXAMPLE

- **The problem is concerned with the purchase of fabric by a fashion designer**
- **The two choices offered by textile suppliers are:**

  supplier 1: ***fixed price – constant* 2 *$/yd***

  supplier 2: ***variable price dependent on quantity at***

  2.10 *$/yd for the first* 20,000 *yd;*

  1.90 *$/yd for the next* 10,000 *yd;*

  1.70 *$/yd for the next* 10,000 *yd;*

  1.50 *$/yd thereafter*

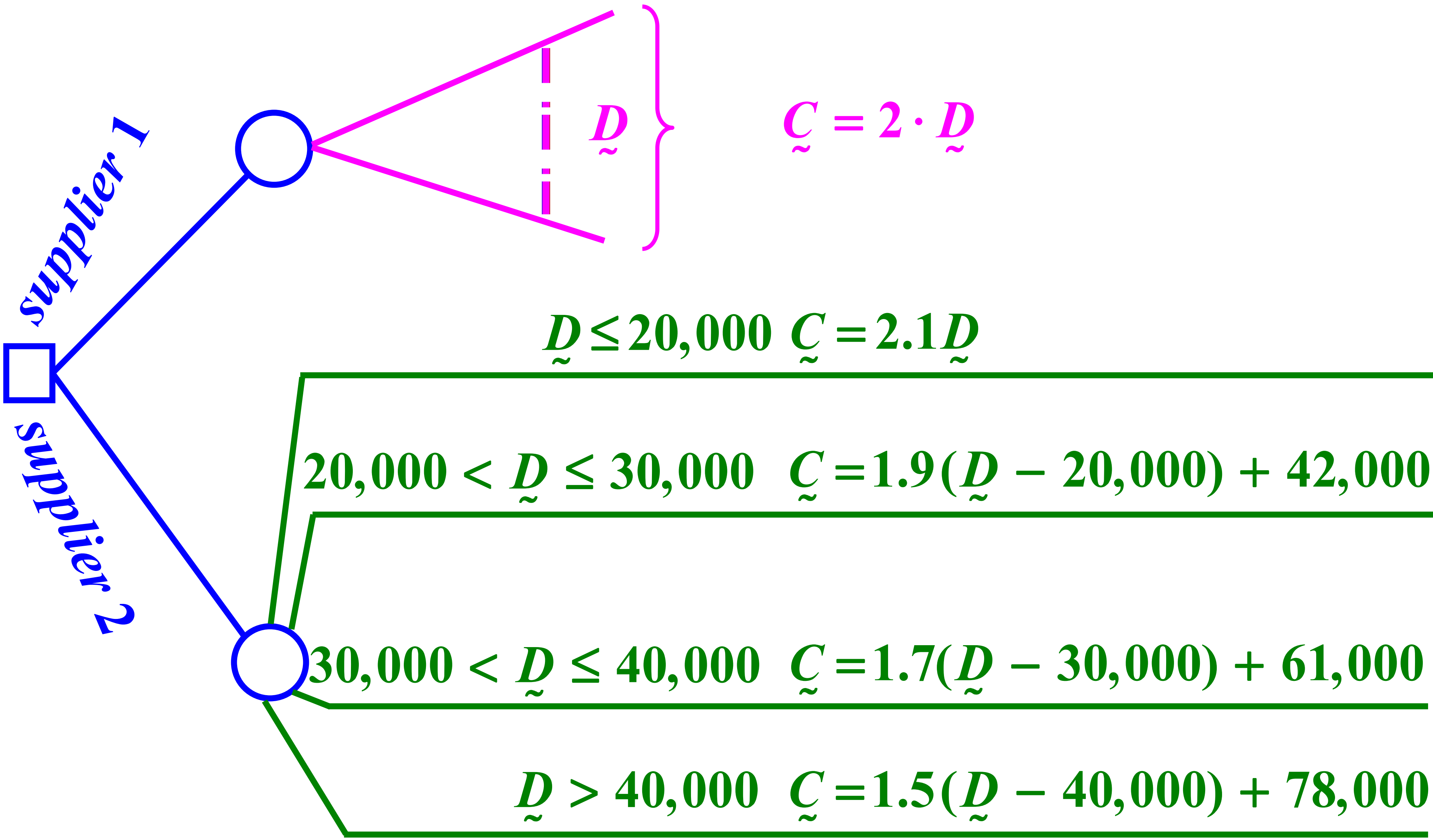
SIMULATION EXAMPLE

- The purchaser is uncertain about the demand $\underset{\sim}{D}$ but determines an appropriate model is:

$$\underset{\sim}{D} \sim \mathcal{N}(25,000\ yd, 5,000\ yd)$$

- The decision may be represented in form of the following decision branches:

SIMULATION EXAMPLE



SIMULATION EXAMPLE

- Supplier 1 has a simple linear cost function $\underset{\sim}{C}$
- Supplier 2 has a more complicated scheme to evaluate costs: in effect, the range of the demand and the corresponding probability for $\underset{\sim}{D}$ to be in a particular segment of the range must be known, as well as the expected value of $\underset{\sim}{D}$ for each range

SIMULATION EXAMPLE

- **We simulate the situation in the decision tree by "*drawing multiple samples from the appropriate population*"**
- **We systematically tabulate the results and evaluate the required statistics**
- **The algorithm for the simulation consists of a few simple steps which are repeated until an *appropriatly sized* sample is constructed**

BASIC ALGORITHM

Step 0 : store the distribution $\mathcal{N}\left(25{,}000,\ 5{,}000\right)$; determine $\bar{k}$, the maximum number of draws; set $k = 0$

Step 1 : if $k > \bar{k}$, stop; else set $k = k + 1$

Step 2 : draw a random sample from the normal distribution $\mathcal{N}\left(25{,}000,\ 5{,}000\right)$

Step 3 : evaluate the outcomes on both branches; enter each outcome into the data base and return to Step 1

SIMULATION EXAMPLE

- Application of the algorithm allows the determi–nation of the histogram of the cost figures and then the evaluation of the expected costs
- For the assumed demand, for supplier 1, we have the straightforward case of

$$\mu_{\underset{\sim}{C}} = E\{\underset{\sim}{C}\} = 2 \cdot E\{\underset{\sim}{D}\} = 50,000$$

and

SIMULATION EXAMPLE

$$\sigma_{\underset{\sim}{C}} = 10,000$$

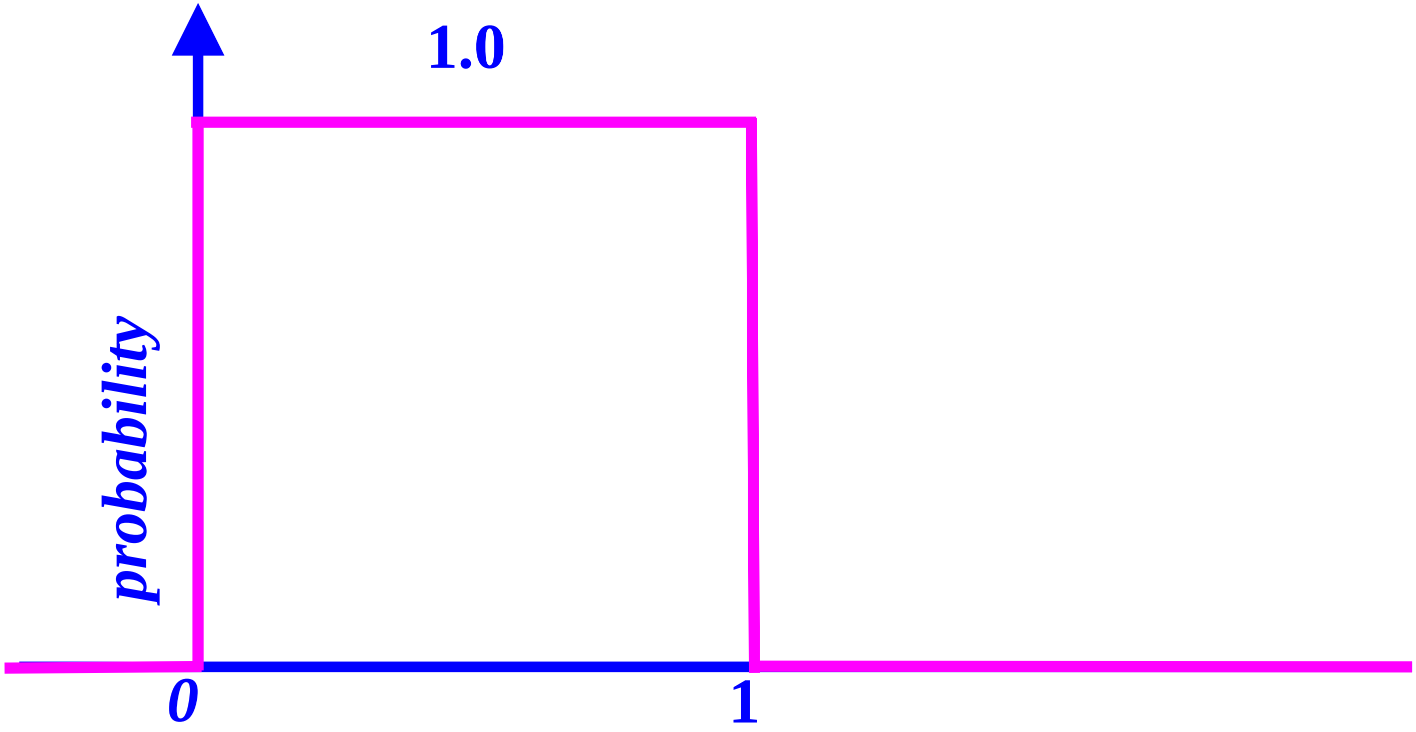
and the use of the algorithm may be bypassed

- For the supplier 2, the algorithm is applied for the $\bar{k}$ random draws
- The actual simulation is an exercise left to the reader

GENERATION OF RANDOM DRAWS

- A key issue is the generation of random draws for which we need a random number generator
- There are various random number generator algorithms
- One natural scheme is based on the use of a uniformly distributed *r.v.* between *0* and 1

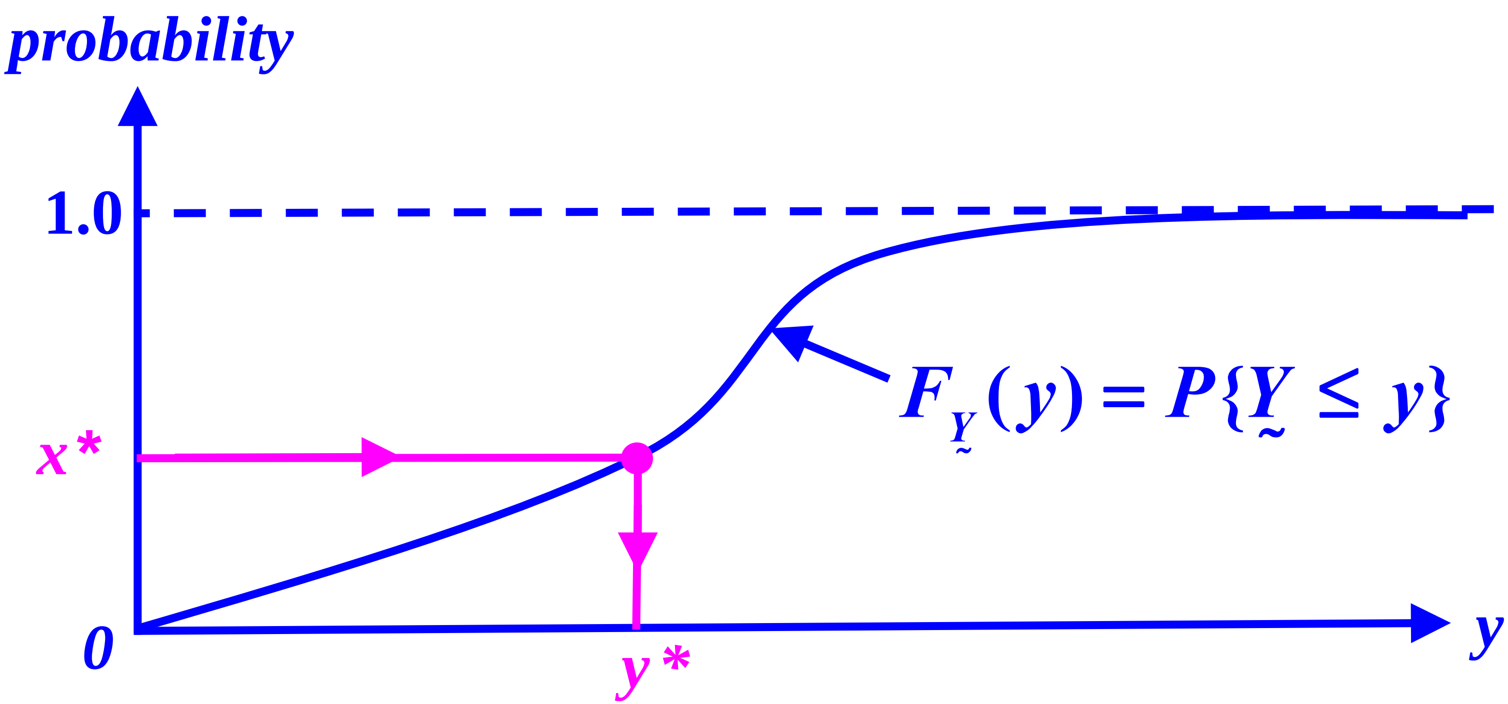
GENERATION OF RANDOM DRAWS



$$\underset{\sim}{X} = \begin{cases} x \in [0,1] \ \textit{with probability} \ \ 1 \\ x \notin [0,1] \ \textit{with probability} \ \ 0 \end{cases}$$

GENERATION OF RANDOM DRAWS

- **We draw a random value of x, say x^* and work through the *c.d.f.* $F_{\underset{\sim}{Y}}(y)$ to get the value y^* of the *r.v.* $\underset{\sim}{Y}$ with $F_{\underset{\sim}{Y}}(y^*) = x^*$**



SOFT PRETZEL EXAMPLE

- The market size is unknown but we assume that the market size is a normally distributed *r.v.* with

$$\underset{\sim}{S} \sim \mathcal{N}\left(\underbrace{100,000}_{\mu_{\underset{\sim}{S}}}, \underbrace{10,000}_{\sigma_{\underset{\sim}{S}}}\right)$$

- We are interested in determining the fraction $\underset{\sim}{F}$ of the market the new company is able to capture
- We model the distribution of $\underset{\sim}{F}$ using the discrete distribution tabulated below:

SOFT PRETZEL EXAMPLE

$\underset{\sim}{F} = x\%$	$P\{\underset{\sim}{F} = x\}$
16	0.15
19	0.35
25	0.35
28	0.15

SOFT PRETZEL EXAMPLE

- Sales price of a pretzel is $\$\ 0.50$
- Variable costs $\underset{\sim}{V}$ are represented by a uniformly distributed *r.v.* in the range $[0.08\ ,\ 0.12]\ \$/pretzel$
- Fixed costs $\underset{\sim}{C}$ are also random
- The contributions to profits are given by

$$\underset{\sim}{\Pi} = (\underset{\sim}{S} \cdot \underset{\sim}{F}) \cdot (0.5 - \underset{\sim}{V}) - \underset{\sim}{C}$$

  and may be evaluated via simulation
- We can use simulation to approximate $F_{\underset{\sim}{\Pi}}(\bullet)$

MANUFACTURING CASE STUDY

- The selection of one of two manufacturing *processes* based on net present value (*NPV*) using a 3 – year horizon – the current year *0* plus the next two years 1 and 2 – and a 10 % discount rate
- The *process* is used to manufacture a product whose sale price is 8 *$/unit*

PROCESS 1 DESCRIPTION

- **This *process* uses the current machinery for manufacturing**

- **The annual fixed costs are $\$12{,}000$**

- **The yearly variable costs are represented by the *r.v.***

$$\underset{\sim}{V}_i \sim \mathcal{N}(4, 0.4) \qquad i = 0, 1, 2$$

PROCESS 1 DESCRIPTION

- Machine in the *process* can fail randomly and the number failures $\underset{\sim}{Z}_i$ in year $i = 0, 1, 2$ is a *r.v.* with

$$\underset{\sim}{Z}_i \sim Poisson\left(m = 4\right) \qquad\qquad i = 0, 1, 2$$

- Each failure incurs constant costs of $ 8,000 over the 3-year period

- Total costs are the sum of $\underset{\sim}{V}_i$ and $8,000\ \underset{\sim}{Z}_i$

PROCESS 1: SALES FORECAST UNCERTAINTY DATA

current year $i = 0$		*next year* $i = 1$		*year after next* $i = 2$	
d_0	$P\{\underset{\sim}{D}_0 = d_0\}$	d_1	$P\{\underset{\sim}{D}_1 = d_1\}$	d_2	$P\{\underset{\sim}{D}_2 = d_2\}$
11,000	0.2	8,000	0.2	4,000	0.1
16,000	0.6	19,000	0.4	21,000	0.5
21,000	0.2	27,000	0.4	37,000	0.4

PROCESS 2: DESCRIPTION

- *Process* 2 involves an investment of $\$60{,}000$ paid in cash to buy the new equipment and doing away with the worthless current machinery; the fixed costs of $\$12{,}000$ per year remain unchanged
- The yearly variable costs $\underset{\sim}{V}_i$

$$\underset{\sim}{V}_i \sim \mathcal{N}\left(\$3.50,\ \$1.0\right) \qquad i = 0, 1, 2$$

- The number of machine failures $\underset{\sim}{Z}_i$ for year

$$\underset{\sim}{Z}_i \sim Poisson\ \left(m = 3\right) \qquad i = 0, 1, 2$$

and the costs per failure are $\$6{,}000$

PROCESS 1: SALES FORECAST UNCERTAINTY DATA

current year $i = 0$		next year $i = 1$		year after next $i = 2$	
d_0	$P\{\underset{\sim}{D}_0 = d_0\}$	d_1	$P\{\underset{\sim}{D}_1 = d_1\}$	d_2	$P\{\underset{\sim}{D}_2 = d_2\}$
14,000	0.3	12,000	0.36	9,000	0.4
19,000	0.4	23,000	0.36	26,000	0.1
24,000	0.3	31,000	0.28	42,000	0.5

NET PROFITS

- The net profits $\underset{\sim}{\pi}_i$ each year are a function

$$\underset{\sim}{\pi}_i = f\left(\underset{\sim}{D}_i, \underset{\sim}{V}_i, \underset{\sim}{Z}_i\right) \qquad\qquad i = 0, 1, 2$$

- While for each *process*, the $F_{\underset{\sim}{\pi}_i}(\cdot)$ approximation requires the evaluation of all the possible outcomes, both $E\left\{\underset{\sim}{\pi}_i\right\}$ and $var\left\{\underset{\sim}{\pi}_i\right\}$ may be estimated by simulation by drawing an appropriate number of samples from the underlying distribution

NPV

- The *NPV* of these profits needs to be assessed and expressed in terms of the current *year 0 dollars*
- The profits are collected at the end of each year or equivalently, at the beginning of the following year
- We use the $d = 10\,\%$ discount factor to express the $var\left\{ \underset{\sim}{\pi}_{\,i} \right\}$ in year *0* (*current*) *dollars*

NPV

- We can evaluate for *processes* 1 and 2 the profits for each year; we use superscript to denote the *process*

$$process\ 1: \quad \underset{\sim}{\pi}_i^{1} = 8\underset{\sim}{D}_i - \underset{\sim}{D}_i \underset{\sim}{V}_i - 8,000\underset{\sim}{Z}_i - 12,000$$

$$i = 0, 1, 2$$

$$process\ 2: \quad \underset{\sim}{\pi}_i^{2} = 8\underset{\sim}{D}_i - \underset{\sim}{D}_i \underset{\sim}{V}_i - 6,000\underset{\sim}{Z}_i - 12,000$$

and we also need to account for the *$* 60,000 *investment* in year *0* for *process* 2

NPV

- The *NPV* evaluation then is stated as the *r.v.*

$$\underset{\sim}{\Pi}^{1} = \sum_{i=0}^{2} \underset{\sim}{\pi}_{i}^{1} \left(1.1\right)^{-\left(i+1\right)}$$

and

evaluated in year 0 dollars

$$\underset{\sim}{\Pi}^{2} = -60,000 + \sum_{i=0}^{2} \underset{\sim}{\pi}_{i}^{2} \left(1.1\right)^{-\left(i+1\right)}$$

- Simulation is used to evaluate

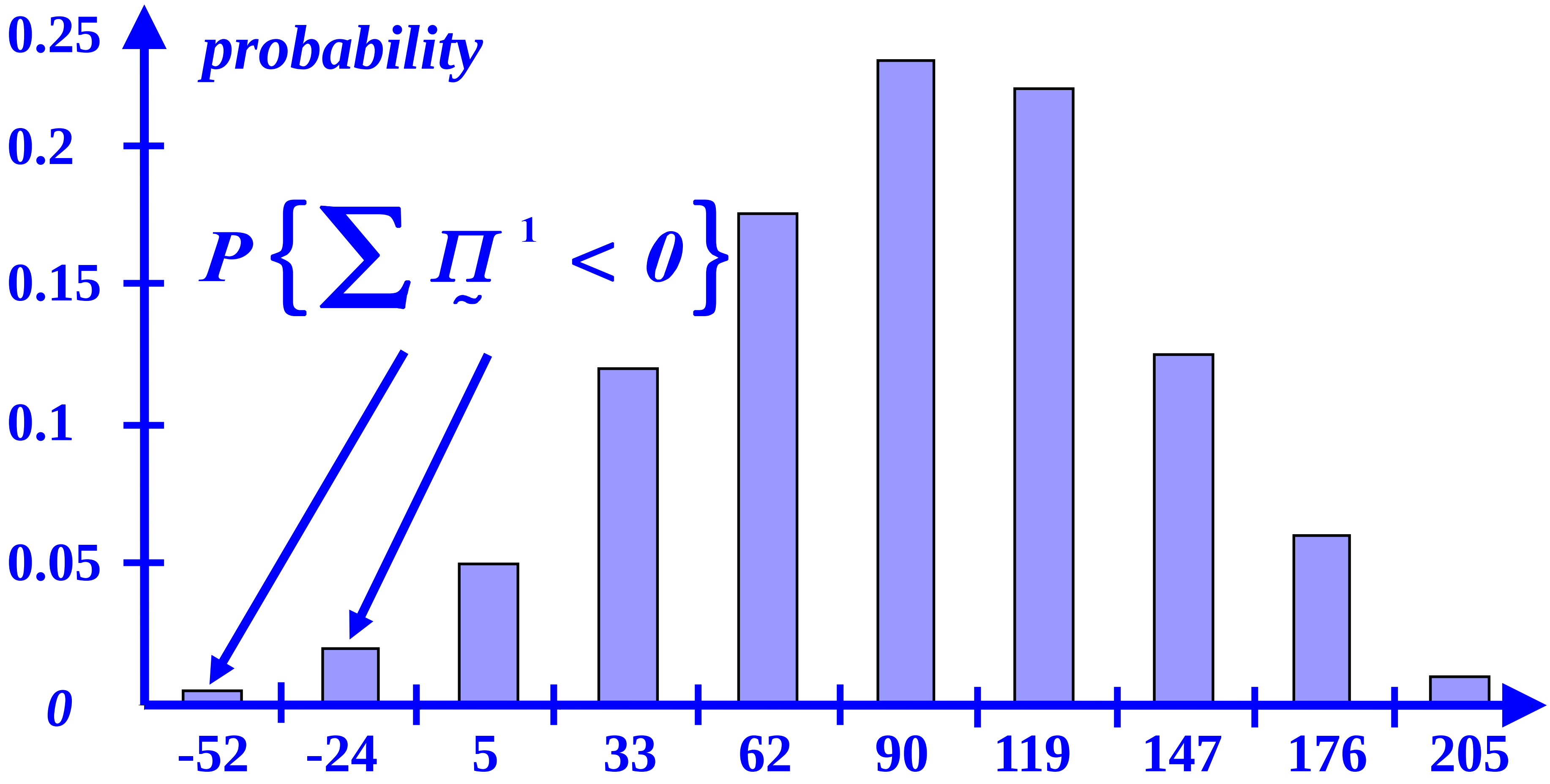
$$NPV^{1} = E\left\{\underset{\sim}{\Pi}^{1}\right\} \quad NPV^{2} = E\left\{\underset{\sim}{\Pi}^{2}\right\}$$

SIMULATION RESULTS

- For a 1,000 replications we obtain

process j	*mean ($)*	*standard deviation ($)*	$P\left\{\sum \underset{\sim}{\Pi}^{j} < 0\right\}$
1	91,160	46,970	0.029
2	110,150	72,300	0.046

SIMULATION RESULTS

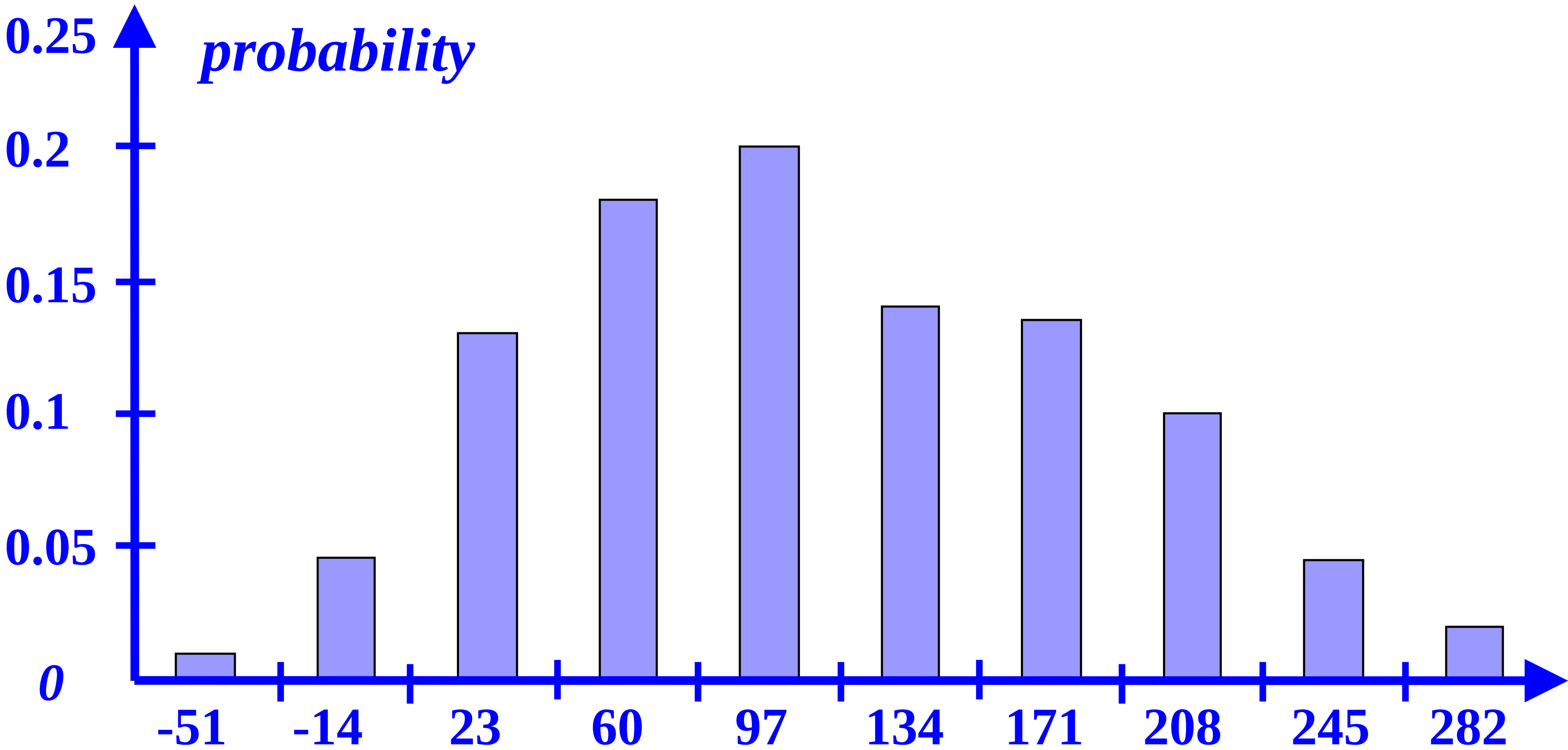


$$NPV^{1} = 91,160$$

$$\sigma = 46,970$$

$$P\left\{ \sum \underset{\sim}{\Pi}^{1} < 0 \right\} = 0.029$$

SIMULATION RESULTS



$$\mathit{NPV}^{2} = 110{,}150$$

$$\sigma = 72{,}300$$

$$P\left\{ \sum \underset{\sim}{\Pi}^{2} < 0 \right\} = 0.046$$

*c.d.f.*s OF THE TWO PROCESSES